Direct Contact Heat Transfer Between Immiscible Fluid Layers in Laminar Flow

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The problem of heat transfer (and, by analogy, mass transfer) between immiscible fluids in laminar parallel flow is analyzed as a conjugated boundary value problem. Exact solutions are developed in terms of known functions by solving for the temperature fields in the two phases separately for constant boundary conditions, then generalizing the results to obtain solutions for the interfacial temperature distribution and the complete temperature field for the conjugated system. The analysis and the results of calculations for heat transfer between two immiscible films flowing down an adiabatic inclined plane are presented. It is shown that axial conduction can be included, and the effects of axial conduction in one stream are examined. The predicted Nusselt numbers are plotted versus axial position for various parameters, and the interfacial temperatures and mixed mean fluid temperatures for the two phases are plotted to elucidate the heat transfer characteristics.

Direct contact heat transfer has attracted some interest in recent years because of its potential application in saline water purification and in other processes where scale formation poses a serious problem. Sideman and his associates (*1* to *7*) have studied a number of direct contact heat transfer techniques, the simplest of which is heat transfer between immiscible films flowing down an inclined plane. Bentwich and Sideman (*1*) analyzed the temperature distribution in such a stratified two-phase laminar flow using a Graetz-type solution, but taking into account the discontinuities in the physical properties at the liquid-liquid interface. They solved the problem for constant thickness films with a free surface for the case of constant wall temperature and constant atmosphere temperature (above the free surface) by rather laborious numerical methods to obtain the first three eigenvalues and eigenfunctions of the appropriate Sturm-Liouville system. In addition they developed approximations to the eigenvalues and eigenfunctions that were of limited accuracy.

Gollan and Sideman (*5*) extended the analysis to the problem of evaporation of the upper film. They considered a number of mathematical models including an exact formulation and simpler approximate models. As they used finite difference techniques they were able to treat the varying thickness of the upper layer. Gollan and Sideman pointed out that a piece-wise Graetz-type solution accounting for the variable thickness of the volatile film can be obtained. Using the solution procedures of Sideman and his associates, however, involves undesirably large amounts of numerical computation. Furthermore, it is well known that to obtain accurate results near the thermal entrance ($x = 0$) it is necessary to have a large number of eigenfunctions using Graetz-type solutions.

It is the purpose of the present paper to show that problems of the type analyzed in (*1*) and (*5*) can be solved in terms of known functions and by procedures that require substantially less machine computation than the numerical techniques previously used. In addition axial conduction in one or both of the phases can be treated with little complication. Although Sideman and his associates did not take axial conduction into account in their analyses it cannot be neglected in some applications that have been suggested, for example, when mercury or another liquid with a high thermal conductivity is used as one of the fluids. A simpler method of analysis should be useful in the evaluation or design of such systems and permits more extensive parametric studies to be performed.

ANALYSIS

The method of analysis used here is to consider each liquid phase separately as a Graetz-type problem, develop solutions for the temperature fields in each phase in a general manner, and match the solutions at an interface by applying appropriate compatibility conditions, that is, continuity of temperature and heat flux at the liquid-liquid interface. In this manner Stein (*8*) analyzed heat transfer with cocurrent flow in double-pipe heat exchangers, Nunge and Gill (*9*) studied laminar counterflow in such heat exchangers and Blanco and Gill (*10*) examined heat transfer in multistream turbulent forced convection systems.

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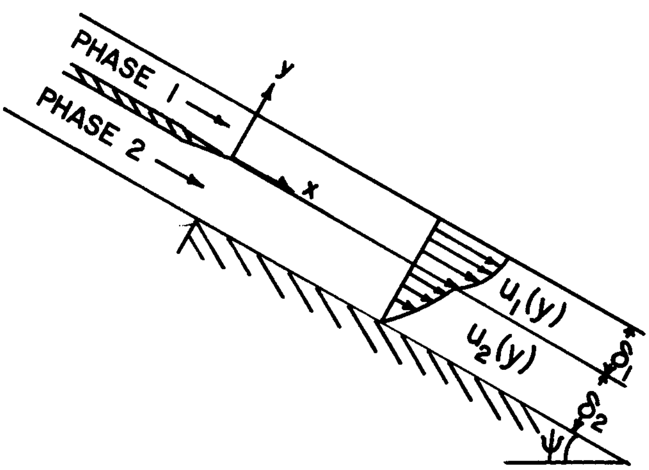

Fig. 1. The system under consideration.

Nunge, Porta and Gill (*11*) extended the analysis to include the effects of axial conduction in the fluid streams. The same general procedure was used by Davis and Gill (*12*) to solve the conjugated boundary value problem associated with heat transfer between a solid boundary and a fluid to determine the effect of axial conduction in the wall on the temperature field in the fluid.

Heat or mass transfer between multilayer rectilinear laminar flows are readily treated by the techniques proposed here. Bird et al. (*13*) considered the velocity profiles for two immiscible liquids in laminar flow between parallel planes, and a somewhat similar system, but with a free surface, is shown in Figure 1. A layer of fluid flows on top of a more dense and immiscible phase which enters at a different temperature than the upper layer. In applications either convective heat transfer to the atmosphere at the free surface or surface evaporation may occur, and the lower boundary may be heated or cooled in some arbitrary way (for the region $x \geqq 0$) or may be an adiabatic surface. Any of these boundary conditions can be treated by the method developed here, but the examples developed in detail are for an adiabatic wall with a constant free surface temperature (the saturation temperature). It is assumed that the liquid layers are of constant thickness, that is, it is considered that the problem of surface evaporation can be treated approximately by a piece-wise solution if a varying film thickness is to be taken into account.

In a mass transfer application the upper and lower surfaces would be bounded by solid walls, and in the absence of any chemical reaction at a wall the no flux condition would apply at the solid boundaries.

VELOCITY DISTRIBUTIONS

Assuming that the fully developed velocity distribution is attained within a short distance from the point at which the two streams are brought into contact the velocity distributions in the two phases are given by

$$U_k(\zeta_k) = \frac{u_k}{u_k^*} = \beta_k + \gamma_k \zeta_k - \zeta_k^2, \quad k = 1, 2 \qquad (1)$$

where

$$\zeta_k = \frac{y}{\delta_k}, \qquad u_k^* = \frac{g \delta_k^2}{2 \nu_k} \sin \psi$$

$$\gamma_1 = 2, \qquad \beta_1 = \frac{\nu_1}{\nu_2} \left(\frac{\delta_2}{\delta_1} \right)^2$$

$$\gamma_2 = 2 \frac{\rho_1}{\rho_2} \frac{\delta_1}{\delta_2}, \qquad \beta_2 = 1 + \gamma_2$$

It should be noted that the velocity distribution of the form of Equation (1) also describes the velocity distributions given by Bird et al. (*13*) for the laminar flow of immiscible liquids between parallel planes. There is some experimental evidence that fully developed free surface flows are rapidly established, for Wilkes and Nedderman (*14*) measured velocity profiles in the entry region of a wetted wall column. Within 0.7 cm. from the downstream edge of an annular distributor the velocity profile shifted from the parabolic profile in the annular region to semiparabolic wave-free flow on the column wall.

For pure liquids even under isothermal conditions the assumptions that the flow is laminar and that fluid-fluid interface is smooth are unlikely to be valid when the inclination of the plate to the horizontal is large because single phase flows of this type can be unstable. The literature on the stability of thin film flow down a vertical or inclined surface is very extensive as indicated by a recent survey of the principal theoretical and experimental studies of the hydrodynamics of such flows by Krantz and Goren (*15*). But surface-active agents are known to stabilize such flows (*16* to *18*). Even with uncontaminated liquids ripple-free flow can exist, for the growth of disturbances can be slow near the neutral stability condition. Stainthorp and Allen (*19*) and Wilkes and Nedderman (*14*) have observed wave-free flow of a thin film down a vertical tube—at least near the inlet.

With multilayer flows other modes of instability are possible. Li (*20*) analyzed the plane Couette flow of three superposed fluid layers, finding that the instability depends upon the modes of waves at the surfaces of discontinuity in the physical properties. It is likely that the stability of multilayer flows of the type considered in Figure 1 depends in a similar manner on the interfacial tension and the viscosity discontinuity at the interface.

There is experimental evidence that free-surface flows of pure components down slightly inclined plates can be ripple-free. Goodridge and Gartside (2) reported ripple-free conditions in their study of mass transfer to a thin liquid film, and Gollan and Sideman (5) reported that stable ripple-free flow was maintained when a thin pentane layer was added on the top of a water layer flowing down a slightly inclined plane.

In addition to the hydrodynamic instabilities associated with isothermal flows heated films can involve instabilities due to density gradients. Frisk (22) observed that at sufficiently high ratios of the Grashof number to the Reynolds number natural convection substantially increases (compared with smooth stable flow) the Nusselt number for heat transfer between a heated wall and a thin liquid film in shear flow under the influence of a turbulent cocurrent gas phase. It should also be noted that he also found that surfactant concentrations as low as 10 ppm damped the wave motion of the horizontal film.

TABLE 1. BOUNDARY AND THERMAL ENTRY CONDITIONS

$$\theta_1(\xi_1, 1) = 0 \qquad \theta_1(0, \zeta_1) = 0$$

$$\frac{\partial \theta_2}{\partial \zeta_2}(\xi_2, -1) = 0 \qquad \theta_2(0, \zeta_2) = 0$$

where, for constant interfacial temperature T_i

$$\theta_1 = \frac{T_1 - T_1^0}{T_i - T_1^0} \qquad \theta_2 = \frac{T_2 - T_2^0}{T_i - T_2^0}$$

Recently Bankoff (*23*) analyzed the stability of liquid flow down a heated inclined plane using classical linear stability analysis. He found that evaporation destabilizes the film, but if the film is cooled a critical Reynolds number is exhibited.

For the illustrative computations discussed below it has been assumed that stable wave-free flow occurs, and the purpose of the computations is to elucidate the parameters that affect the interfacial heat transfer. In any specific application of the present analysis stability considerations are most important, for heat transfer can be greatly affected by instability of the flow.

ENERGY EQUATIONS

Assuming negligible viscous dissipation of energy in the streams, the temperature fields in the two phases are described by the convective diffusion equation

$$U_k(\zeta_k)\,\frac{\partial\theta_k}{\partial\xi_k} = \frac{\partial^2\theta_k}{\partial\zeta_k^2} + \frac{1}{Pe_k^2}\,\frac{\partial^2\theta_k}{\partial\xi_k^2}, \quad k = 1, 2 \tag{2}$$

where

$$Pe_k = \frac{\delta_k u_k^*}{\alpha_k}, \quad \xi_k = \frac{x}{\delta_k Pe_k} \quad \text{and}$$

$\theta_k = \theta_k(\xi_k, \zeta_k)$ is the dimensionless temperature defined below.

The last term in the equation, the axial conduction term, may be neglected for sufficiently large Péclet numbers. As we shall consider applications where axial conduction is negligible and cases in which the axial conduction is significant in at least one phase we shall include the axial conduction term throughout the analysis. Hsu (*24*) has shown how axial conduction can be included in the solution of the classical Graetz problem without introducing great complexity, and we shall use Hsu's technique.

BOUNDARY CONDITIONS

When axial conduction occurs it is not rigorously correct to specify the thermal entry condition a priori, for axial conduction can be expected to distort the temperature profile upstream of the point at which the fluids are brought into contact. Agrawal (*25*) examined heat transfer in laminar flow between parallel plates at small Péclet numbers for a step change in the wall temperature. For $Pe = 1$ the temperature profile at the point ($x = 0$) where the temperature jump occurred is considerably distorted from the uniform temperature profile considered to occur far upstream from the discontinuity. Even for $Pe = 1$, however, the temperature profile well downstream of $x = 0$ is independent of the temperature distribution at $x = 0$.

For simplicity we shall assume that even when axial conduction is significant the temperature profiles at the point of initial contact of the streams can be specified a priori as uniform in each phase. For $Pe < 100$ this simplification leads to results of dubious accuracy near $x = 0$, but the accuracy can be expected to improve with increasing x. Furthermore, near $x = 0$ the flow is actually still developing, so the analysis is not rigorous in this hydrodynamic entrance region.

THE SOLUTION FOR THE INDIVIDUAL PHASES

The temperature field in each phase can be obtained by solving Equation (2) subject to the appropriate boundary condition at either the free surface or the wall together with an assumed constant boundary condition at the liquid-liquid interface. The solutions for an arbitrarily varying interfacial temperature are then generated by applying Duhamel's theorem, and the campatibility conditions at the interface are applied to give an integral equation which is solved to determine the interfacial temperature distribution. The compatibility conditions are:

$$T_1(x, 0) = T_2(x, 0) \tag{i}$$

$$k_1\,\frac{\partial T_1}{\partial y}\,(x, 0) = k_2\,\frac{\partial T_2}{\partial y}\,(x, 0) \tag{ii}$$

Once the interfacial temperature distribution is known the complete temperature fields can be calculated and the heat transfer characteristics can be determined. In a mass transfer application condition (i) would be replaced by the appropriate equilibrium condition (say, in terms of a partition coefficient) at the liquid-liquid interface.

The convective diffusion equation has the same form for each phase, which is

$$(-\,\zeta^2 + \gamma\zeta + \beta)\,\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\zeta^2} + \frac{1}{Pe^2}\,\frac{\partial^2\theta}{\partial\xi^2} \tag{3}$$

We have dropped the subscripts, for it is only necessary to present the solution details for one of the phases. The solution of Equation (3) has been discussed in some detail by Davis elsewhere (*28*). Equation (3) has a solution of the form

$$\theta(\xi, \zeta) = \theta_\infty + \sum_{m=1}^{\infty} A_m e^{-\lambda_m^2\xi}\,\phi_m(\zeta) \tag{4}$$

where θ_∞ is the asymptotic or fully developed temperature profile, and the eigenvalues λ_m and the eigenfunctions $\phi_m(\zeta)$ satisfy appropriate homogeneous boundary conditions. The eigenfunctions are given by

$$\phi_m(\zeta) = e^{-z/2}\left[k_{1_m} M\left(a, \frac{1}{2}, z\right) + k_{2_m} z^{1/2} M\left(a + \frac{1}{2}, \frac{3}{2}, z\right)\right] \tag{5}$$

where

$$z = \lambda_m\left(\zeta - \frac{\gamma}{2}\right)^2,$$

$$a = \frac{1}{4} - \frac{[\gamma^2 + 4(\beta + \lambda_m^2/Pe^2)]\lambda_m}{16},$$

and $M(a, b, z)$ is the confluent hypergeometric function (*26*). As the coefficients A_m in Equation (4) are yet to be determined we may arbitrarily put either k_{1_m} or k_{2_m} equal to unity. The other constants k_{2_m} or k_{1_m} and the eigenvalues λ_m are obtained by applying the boundary conditions to Equation (5).

Finally the expansion coefficients A_m are obtained by applying the thermal entry condition, using Hsu's modified orthogonality property to give

$$A_m = \frac{\displaystyle\int_0^{\pm 1} (-\,\zeta^2 + \gamma\zeta + \beta + 2\lambda_m^2/Pe^2)\,\theta(0, \zeta)\,\phi_m(\zeta)\,d\zeta}{\displaystyle\int_0^{\pm 1} (-\,\zeta^2 + \gamma\zeta + \beta + 2\lambda_m^2/Pe^2)\,\phi_m^2(\zeta)\;d\zeta} \tag{6}$$

The upper limit of the integrals is $+1$ for phase 1 and -1 for phase 2. When the Péclet number is sufficiently large

Equation (6) may be replaced by the usual Sturm-Liouville result, that is,

$$A_m = \frac{2}{\lambda_m} \frac{[\theta(0,\zeta)\phi_m' - \phi_m\theta'(0,\zeta)]_0^{\pm 1}}{\left[\phi_m \left(\frac{\partial \phi_m}{\partial \lambda_m} \right)' - \phi_m' \frac{\partial \phi_m}{\partial \lambda_m} \right]_0^{\pm 1}} \tag{7}$$

where the prime refers to differentiation with respect to ζ.

This procedure, involving the confluent hypergeometric function, has very substantial computational advantages over solving the appropriate Sturm-Liouville problem numerically.

THE SINGLE-PHASE SOLUTIONS

For a constant interfacial temperature the asymptotic temperature profile for the upper phase is given by $\theta_\infty = 1 - \zeta$. The eigenfunctions must satisfy the homogeneous boundary conditions, $\phi_m(0) = \phi_m(1) = 0$. Applying the boundary condition at $\zeta = 1$ we obtain $k_{1_m} = 0$. Putting $k_{2_m} = 1$ the eigenfunctions are given by

$$\phi_m(\zeta) = e^{-z/2} z^{1/2} M\left(a + \frac{1}{2}, \frac{3}{2}, z \right) \tag{8}$$

where

$$z = \lambda_m(\zeta - 1)^2 \quad \text{and} \quad a = \frac{1 - (1 + \beta + \lambda_m^2/Pe^2)\lambda_m}{4}$$

The eigenvalues are obtained by applying the boundary condition at $\zeta = 0$. A transcendental equation in λ_m is obtained which is solved by numerical methods to generate the eigenvalues, that is,

$$M\left(a + \frac{1}{2}, \frac{3}{2}, \lambda_m \right) = 0 \tag{9}$$

As the confluent hypergeometric function can be calculated with high accuracy the eigenvalues can be calculated with high accuracy. Using asymptotic eigenvalues as the first approximation in Equation (9), or its equivalent for the other phase and for other boundary conditions, the eigenvalues are readily obtained by a Newton-Raphson technique. Eigenvalues computed in this way are presented in Table 2 for the boundary conditions and parameters of interest here and more complete tables of eigenvalues are provided in reference 27.

Finally, the expansion coefficients for the solution of the constant interfacial boundary condition for phase 1 are given by Equation (6) for low Péclet numbers, or by Equation (10) for large Pe.

$$A_m = \frac{2}{\lambda_m \left(\frac{\partial \phi_m}{\partial \lambda_m} \right)\Big|_{\zeta=0}} \tag{10}$$

The derivative with respect to the eigenvalue is obtained numerically.

The solution for the temperature field in phase 2 for a constant interfacial temperature is obtained in the same manner, and the solution has the form

$$\theta(\xi, \zeta) = 1 + \sum_{n=1}^{\infty} B_n e^{-\Lambda_n^2 \xi} \Phi_n(\zeta) \tag{11}$$

TABLE 2. A PARTIAL LIST OF EIGENVALUES FOR THE EXAMPLES TREATED

	System 1A		System 1B	
n	λ_n	Λ_n	λ_n	Λ_n
1	0.6201153	1.96522	1.067305	1.76050
2	1.2402307	5.40772	2.134611	4.81296
3	1.8603461	8.83856	3.201917	7.86137
4	2.4804614	12.26776	4.269222	10.90890
5	3.1005768	15.69608	5.336528	13.95605
6	3.7206922	19.12395	6.403834	17.00286
7	4.3408076	22.55155	7.471140	20.04976
8	4.9609229	25.97897	8.538445	23.09850
9	5.5810383	29.40627	9.605751	26.14308
10	6.2011537	32.83339	10.673057	29.18966

	System 2A and 2B	System 2A	System 2B
n	λ_n	Λ_n	Λ_n (Neglecting axial conduction)
1	1.455034	2.07949	2.15995
2	2.910069	4.98467	5.98541
3	4.365103	7.16057	9.79173
4	5.820138	8.94758	13.59422
5	7.275172	10.48491	17.39521
6	8.730207	11.84697	21.19543
7	10.185241	13.07934	24.99520
8	11.640276	14.21163	28.79468
9	13.095310	15.26403	32.59397
10	14.550345	16.25094	36.39311

where the Λ_n and $\Phi_n(\zeta)$ satisfy the boundary conditions $\Phi_n(0) = \frac{\partial \Phi_n}{\partial \zeta}(-1) = 0$. Equation (5) gives the form of the eigenfunctions, but for phase 2 we have $k_{1_n} = 1$ and

$$k_{2n} = -\frac{M\left(a, \frac{1}{2}, \Lambda_n\gamma_2^2/4 \right)}{\frac{\Lambda_n^{1/2}\gamma_2}{2} M\left(a + \frac{1}{2}, \frac{3}{2}, \frac{\Lambda_n\gamma_2^2}{4} \right)}, \tag{12}$$

and the eigenvalues are obtained as the zeros of Equation (13)

$$2aM\left(a+1, \frac{3}{2}, z(-1) \right) - \frac{1}{2} M\left(a, \frac{1}{2}, z(-1) \right) + k_{2n} \left\{ \frac{1}{2}[(z(-1))^{-1/2} - (z(-1))^{1/2}] \right. M\left(a + \frac{1}{2}, \frac{3}{2}, z(-1) \right) + \frac{2}{3}(z(-1))^{1/2} \left(a + \frac{1}{2} \right) M\left(a + \frac{3}{2}, \frac{5}{2}, z(-1) \right) = 0 \tag{13}$$

where $z(-1) = \Lambda_n \left(1 + \frac{\gamma_2}{2} \right)^2$.

The expansion coefficients B_n are obtained in the same way as the A_m to complete the solution for phase 2. With the solutions for the two phases for a constant interfacial temperature in hand we may proceed to obtain the solutions for arbitrarily varying $T_i = T_i(x)$, where T_i is the interfacial temperature.

Applying Duhamel's theorem to the solutions of the constant boundary condition problems we obtain the solutions for arbitrarily varying T_i. Thus

$$T_1(x, y) = \frac{\partial}{\partial x} \int_0^x \left\{ T_1^0 + [T_i(x') - T_1^0] \right.$$

$$\left[1 - \frac{y}{\delta_1} + \sum_{m=1}^{\infty} A_m \exp\left[-\epsilon_m (x - x')\right] \phi_m\left(\frac{y}{\delta_1}\right)\right]\Bigg\} \, dx' \quad (14)$$

and

$$T_2(x, y) = \frac{\partial}{\partial x} \int_0^x \Bigg\{ T_2^0 + [T_i(x') - T_2^0] \left[1 + \sum_{n=1}^{\infty} B_n \exp\left[-E_n(x - x')\right] \Phi_n\left(\frac{y}{\delta_2}\right)\right]\Bigg\} \, dx' \quad (15)$$

where x' is a dummy variable, $\epsilon_m = \frac{\lambda_m^2}{Pe_1 \delta_1}$ and $E_n = \frac{\Lambda_n^2}{Pe_2 \delta_2}$. The temperature gradients are given by

$$\frac{\partial T_1}{\partial y}(x, y) = \frac{1}{\delta_1} \frac{\partial}{\partial x} \int_0^x \Bigg\{ T_1^0 + [T_i(x') - T_1^0] \left[-1 + \sum_{m=1}^{\infty} A_m \exp\left[-\epsilon_m (x - x')\right] \phi_m'\right]\Bigg\} \, dx' \quad (16)$$

and

$$\frac{\partial T_2}{\partial y}(x, y) = \frac{1}{\delta_2} \frac{\partial}{\partial x} \int_0^x \Bigg\{ T_2^0 + [T_i(x') - T_2^0] \left[\sum_{n=1}^{\infty} B_n \exp\left[-E_n (x - x')\right] \Phi_m'\right]\Bigg\} \, dx' \quad (17)$$

where ϕ_m' and Φ_n' are derivatives with respect to ζ_1 and ζ_2, respectively.

THE SOLUTION OF THE CONJUGATED PROBLEM

To obtain the temperature fields in the two phases for the true interfacial temperature distribution it is necessary to solve for $T_i(x)$. This is accomplished as follows. Applying the compatibility condition (ii) (continuous flux at the interface) using Equations (16) and (17) we have an integral equation to solve for $T_i(x)$.

Applying Leibnitz's rule to Equations (16) and (17) and substituting the results in the compatibility condition we obtain

$$\begin{aligned} T_1^0 + [T_i(x) - T_1^0] &\left[-1 + \sum_{m=1}^{\infty} A_m \phi_m'(0)\right] \\ &- \int_0^x \Bigg\{ [T_i(x') - T_1^0] \left[\sum_{m=1}^{\infty} A_m \epsilon_m \exp\left[-\epsilon_m (x - x')\right] \phi_m'(0)\right]\Bigg\} \, dx' \\ &= \omega T_2^0 + \omega\, [T_i(x) - T_2^0] \left[\sum_{n=1}^{\infty} B_n \Phi_n'(0)\right] \\ &- \omega \int_0^x \Bigg\{ [T_i(x') - T_2^0] \left[\sum_{n=1}^{\infty} B_n E_n \exp\left[-E_n (x - x')\right] \Phi_n'(0)\right]\Bigg\} \, dx' \end{aligned} \quad (18)$$

where

$$\omega = \frac{k_2}{k_1} \frac{\delta_1}{\delta_2}$$

Taking the Laplace transform of each side of the equation, using the convolution theorem for the Laplace transforms of the integrals, and rearranging, we obtain

$$\overline{T_i}(s) P + \frac{T_1^0}{s}(1 - P) = \frac{\omega T_2^0}{s}(1 - Q) + \omega \overline{T_i}(s) Q \quad (19)$$

where $\overline{T_i}(s)$ is the Laplace transform of $T_i(x)$

$$P = -1 + \sum_{m=1}^{\infty} A_m \left(1 - \frac{\epsilon_m}{s + \epsilon_m}\right) \phi_m'(0)$$

and

$$Q = \sum_{n=1}^{\infty} B_n \left(1 - \frac{E_n}{s + E_n}\right) \Phi_n'(0)$$

Solving for $\overline{T_i}(s)$ we have

$$\overline{T_i}(s) = \frac{\omega T_2^0 (1 - Q) + T_1^0 (1 - P)}{s(P - \omega Q)} \quad (20)$$

Equation (20) has the form

$$T_i(s) = \frac{p(s)}{q(s)} \quad (21)$$

where $p(s)$ and $q(s)$ are polynomials in s, and $q(s)$ is of higher degree than $p(s)$. The inverse of Equation (21) may be obtained by using the Heaviside expansion formula to give

$$T_i(x) = \sum_{m=1}^{M} C_j e^{-\sigma_j x}, \quad \sigma_j > 0 \quad (22)$$

where $C_j = p(-\sigma_j)/q'(-\sigma_j)$, $-\sigma_j$ are the zeros of the denominator polynomial $q(s)$ and q' is the derivative with respect to s. The zeros of $q(s)$ may be found by numerical methods. To obtain accurate values of the C_j it has been found necessary to include a large number of terms in the polynomials $p(s)$ and $q(s)$. Frequently forty terms or more in the series P and Q are required for reasonable accuracy.

Once the interfacial temperature distribution is known the temperature fields $T_1(x, y)$ and $T_2(x, y)$ can be obtained from Equation (14) and (15). The complete temperature distributions are usually of less interest than the heat transfer rate, so it is convenient to describe the interfacial heat transfer characteristics by defining an interfacial Nusselt number Nu in terms of the local heat flux at the interface and the difference in the mixed mean temperatures of the two phases. Thus

$$\begin{aligned} Nu &= -\frac{\delta_2}{(T_{2m} - T_{1m})} \left.\frac{\partial T_2}{\partial y}\right|_{y=0} \\ &= -\frac{\delta_1}{(T_{2m} - T_{1m})} \left.\frac{\partial T_1}{\partial y}\right|_{y=0} \end{aligned} \quad (23)$$

where the mixed mean temperature is defined by

$$T_{km}(x) = \frac{\int_0^{\pm 1} U_k(y) T_k(x, y) \, dy}{\int_0^{\pm 1} U_k(y) \, dy}, \quad k = 1, 2 \quad (24)$$

The upper limit of the integration is +1 for phase 1 and −1 for phase 2.

Substituting the solution for $T_i(x)$ in Equation (16) and using the result in the definition of the Nusselt number we obtain, after some manipulation

$$Nu = \frac{T_1^0}{(T_{2m} - T_{1m})} \sum_{m=1}^{\infty} A_m \phi_m'(0) e^{-\epsilon_m x}$$

$$- \frac{2T_1^0}{(T_{2m} - T_{1m})} - \frac{1}{(T_{2m} - T_{1m})} \sum_{j=1}^{M} \sum_{m=1}^{\infty} C_j A_m \phi_m'(0)$$

$$\frac{(\epsilon_m e^{-\epsilon_m x} - \sigma_j e^{-\sigma_j x})}{(\epsilon_m - \sigma_j)} \qquad (25)$$

The Nusselt number could also be written in terms of the solution for phase 2, that is,

$$Nu = \frac{T_2^0}{(T_{2m} - T_{1m})} \sum_{n=1}^{\infty} B_n \Phi_n'(0) e^{-E_n x}$$

$$- \frac{1}{(T_{2m} - T_{1m})} \sum_{j=1}^{M} \sum_{n=1}^{\infty} C_j B_n \Phi_n'(0)$$

$$\frac{(E_n e^{-E_n x} - \sigma_j e^{-\sigma_j x})}{(E_n - \sigma_j)} \qquad (26)$$

RESULTS

The solution outlined above is computationally efficient because the confluent hypergeometric function can be computed accurately and the asymptotic approximations discussed in reference 26 can be applied to obtain the higher eigenfunctions. The primary computational difficulty with the present approach is due to the fact that a large number of eigenconstants, eigenvalues and eigenfunctions for each phase are required to compute the polynomials $p(s)$ and $q(s)$ in Equation (21) with sufficient accuracy to make the inversion Equation (22) accurate. For stability of the solution for $T_i(x)$ it is necessary to use more than 25 terms in each series in Equation (18). If the eigenvalues are closely spaced it may be necessary to take in excess of 50 terms to insure stability of the Laplace transform inversion. It is to be noted from Table 2 that the eigenvalues for phase 1 in most of the examples treated here are more closely spaced than for phase 2. Consequently more terms were required for phase 1 than for phase 2.

The zeros of $q(s)$ were obtained by a direct search technique. In general $q(s)$ has discontinuities and the zeros are not regularly spaced, but in most of the computations whose results are reported here the higher zeros were so widely spaced that the series in Equation (22) converged very rapidly. Usually only 4 or 5 terms were required at that point.

Before proceeding to the results of the parametric study we can draw some qualitative information about the heat transfer characteristics from the above solution. An examination of Equation (18) indicates that the interfacial temperature and the heat transfer characteristics depend on the entry temperatures T_1^0 and T_2^0, the film thicknesses δ_1 and δ_2, the Péclet numbers of the two phases and the parameter ω as well as the fluid dynamics parameters and the boundary conditions. Other parameters being equal, ω is a measure of the ratio of the transverse resistance to heat transfer in phase 1 to that in phase 2. When $\omega >> 1$ the resistance to heat transfer in phase 1 can be expected to be large, and the interfacial temperature should be close to the mixed mean temperature of the hotter fluid (phase 2). When $\omega << 1$ the resistance of phase 2 can be expected to predominate, and the interfacial temperature should approach the mixed mean temperature of phase 1.

The magnitude of ω, however, indicates little about the magnitude of the Nusselt number for the interfacial heat transfer because the Nusselt number depends upon the sum of the resistances to heat transfer in the individual phases. The individual phase resistances depend, in turn, on the convective diffusion associated with the particular phase, characterized by the Péclet number. An examination of either Equation (25) or (26) (recognizing that $\epsilon_m = \lambda_m^2/\delta_1 Pe_1$ and $E_n = \Lambda^2{}_n/\delta_2 Pe_2$) indicates that increasing the Péclet number of either phase tends to increase the interfacial Nusselt number at fixed x because for sufficiently large Péclet numbers A_m, B_n, $\phi_m(0)$ and $\phi_n(0)$ are not functions of Pe, and the exponential terms in the series increase as Pe_1 and Pe_2 increase.

To quantitatively examine the heat transfer characteristics predicted by the above analysis we have performed a parametric study for the wall and free surface boundary conditions of Table 1. As the number of independent parameters is large it is inconvenient and not particularly informative to study the effects of each parameter separately. To examine the effects of the parameters on the heat transfer characteristics we shall consider two systems:

1. An evaporating organic upper phase in contact with an immiscible and hotter dense phase with both phases having a relatively high Péclet number ($Pe_1 = Pe_2 = 500$).
2. A volatile liquid (for example, saline water) in contact with a high density and high thermal conductivity (for example, mercury) lower phase. In this illustration axial conduction in the dense phase is considered.

Figure 2 shows the Nusselt number as a function of the dimensionless axial coordinate ξ_2 for two sets of parameters (system 1A and 1B) that yield rather different heat transfer characteristics. The parameters of particular interest here are indicated on the figure. As in single phase flow the Nusselt number decreases with increasing axial distance, and an asymptotic Nusselt number is attained at some distance from the point of contact of the two phases.

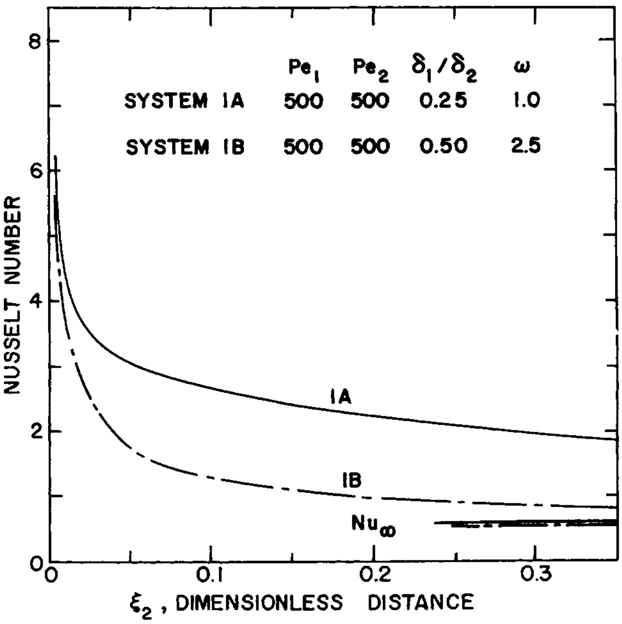

Fig. 2. Interfacial Nusselt number as a function of axial distance for a constant free surface temperature.

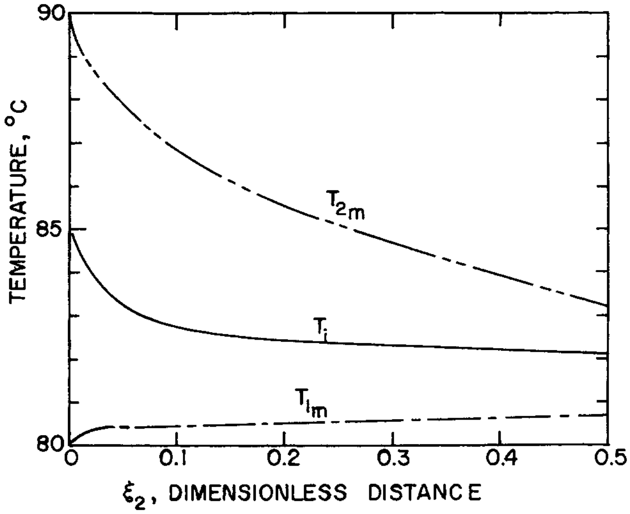

Fig. 3. Axial temperature variations for system 1A.

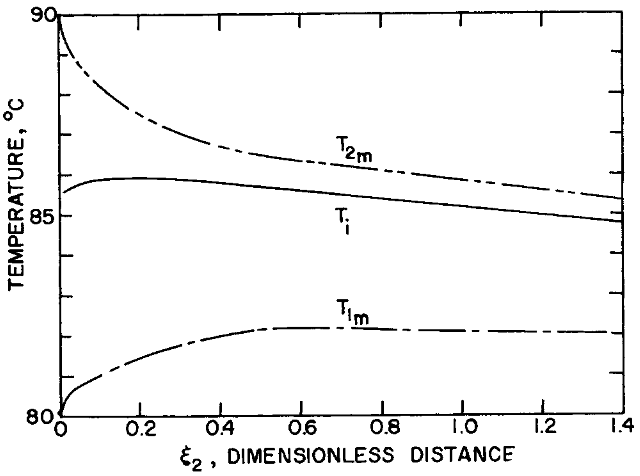

Fig. 4. Axial temperature variations for system 1B.

The essential differences between 1A and 1B are the thickness ratio δ_1/δ_2 and ω.

The interfacial temperature distributions and the mixed mean temperatures provide even more information about the heat transfer characteristics for systems 1A and 1B. For 1A Figure 3 shows that the interfacial temperature decreases monotonically from an inlet value approximately midway between the assumed inlet temperatures of the two liquid layers ($T_1^0 = 80$°C. and $T_2^0 = 90$°C. in these examples) to an asymptotic temperature of 80°C. (beyond the region shown in the figure). The interfacial temperature remains approximately midway between T_{1m} and T_{2m}, which indicates that the resistances to heat transfer are about equal in the two phases. This is to be expected since $\omega = 1$ and $Pe_1 = Pe_2$ for system 1A.

Figure 4 shows a rather different result for system 1B. The interfacial temperature has a maximum at $\xi_2 \approx 0.2$ and is shifted toward T_{2m}, suggesting that phase 1 has the higher resistance to transverse heat transfer. Because of the relatively low Nusselt number for system 1B the asymptotic temperature, 80°C., of the streams is attained at a greater dimensionless distance than for system 1A.

The effects of axial conduction in one phase have been examined, and typical results are shown in Figures 5 and 6. For system 2A the axial conduction term was included in the calculations of the eigenvalues, eigenfunctions and eigenconstants for phase 2, and for 2B the axial conduction term was neglected for both phases. Figure 5 is a comparison between the Nusselt numbers predicted for these two illustrations. The effect of axial conduction is to increase the Nusselt number (in the thermal entry region) over that predicted by neglecting axial conduction, which is consistent with results for single phase flow (*24*). The axial temperature variations of T_i, T_{1m} and T_{2m} show a less intuitively obvious result (comparing results for axial conduction with those for no axial conduction). Figure 6 shows that when axial conduction in phase 2 is neglected the mixed mean temperatures predicted are lower than when axial conduction is included. Heat conduction to the downstream region from the hotter upstream region leads to higher mixed mean temperatures in phase 2 than otherwise predicted even though the heat transfer rate is relatively high because of the larger Nusselt number. The interfacial temperature and T_{1m} are also affected by the axial conduction in the hotter phase. Because of the high thermal conductivity of phase 2 the predominant resistance to transverse heat transfer is in phase 1, and T_i is closer to T_{2m} than to T_{1m}.

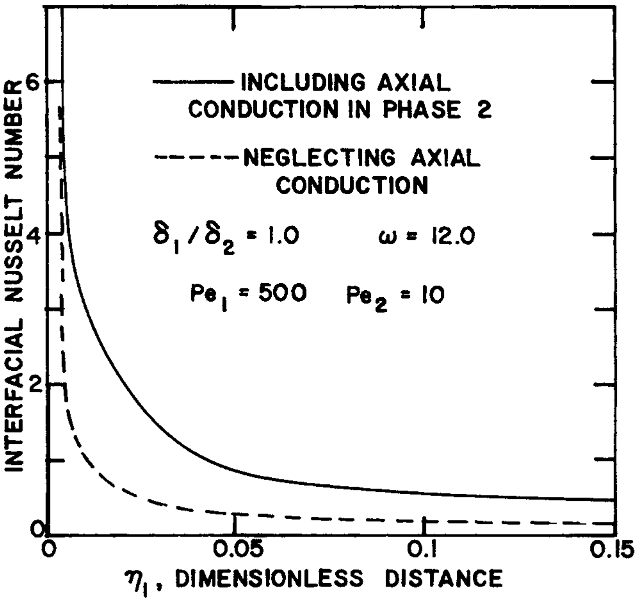

Fig. 5. The effects of axial conduction in phase 2 on the predicted Nusselt numbers.

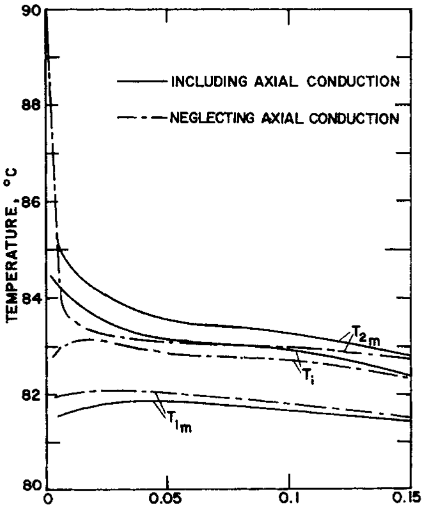

Fig. 6. The effects of axial conduction in phase 2 on the axial temperature variations.

CONCLUSIONS

The problem of direct contact heat transfer (or mass transfer) between immiscible fluid layers in laminar flow has been solved as a conjugated boundary value problem. By expressing the solutions of the single stream problems in terms of known functions the eigenvalues, eigenfunctions and eigenconstants needed for the solution of the conjugated problem are obtained in an efficient and accurate manner. This technique requires far less effort than solving the multi-phase problem as a single-phase Sturm-Liouville problem with discontinuous properties, and a large amount of numerical computation (to obtain eigenvalues and eigenfunctions) is avoided.

Illustrations of typical results of this analysis have been presented as interfacial Nusselt numbers versus axial position for boundary conditions of some interest, and interfacial and mixed mean temperatures have been plotted to elucidate the heat transfer characteristics. There is considerable similarity between single phase heat transfer and and multi-phase direct-contact heat transfer in that the Nusselt number shows the same basic characteristics, but the interfacial temperature in multi-phase systems is a complicated function of the flow rates and physical properties of both phases as well as the inlet temperatures and boundary conditions.

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NOTATION

a, b = parameters in Kummer's equation
A_m = expansion coefficients for phase 1
B_n = expansion coefficients for phase 2
C_j = expansion coefficients for $T_i(x)$
g = gravitational acceleration constant
k = thermal conductivity
$k_{1m}, k_{2m}, k_{1n}, k_{2n}$ = integration constants
$M(a, b, z)$ = the confluent hypergeometric function
Nu = interfacial Nusselt number
$p(s)$ = polynomial in s
P = infinite series defined in the paper
Pe = Péclet number
$q(s)$ = polynomial in s
Q = infinite series defined in the paper
s = Laplace transform variable
T = temperature
$T_i(x)$ = interfacial temperature distribution
T_{k_m} = mixed mean temperature
$\overline{T}(s)$ = Laplace transform of $T_i(x)$
$T_1{}^0, T_2{}^0$ = inlet temperatures
u = velocity distribution
u^* = characteristic velocity
U = dimensionless velocity
W = dependent variable in Kummer's equation
x = axial coordinate
x' = dummy variable
y = transverse coordinate
z = independent variable in Kummer's equation

Greek Letters

α = thermal diffusivity
β, γ = velocity distribution parameters
δ = film thickness
ϵ_m, E_n = eigenvalue parameters
ζ = dimensionless transverse coordinate
θ = dimensionless temperature
θ_∞ = asymptotic temperature distribution
λ_m, Λ_n = eigenvalues
μ = viscosity
$\nu = \mu/\rho$ = kinematic viscosity
ξ, η = dimensionless axial coordinate
ρ = density
σ_j = zeros of the polynomial $q(s)$
ϕ_m, Φ_n = eigenfunctions
ψ = angle of inclination
ω = dimensionless parameter

Subscripts

1 = refers to the upper (less dense) phase
2 = refers to the lower (more dense) phase
i = refers to an interfacial condition
k = refers to the kth phase
s = refers to a surface condition

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